Unique factorisation Domain :- (UFD) Let R be an integral Domain with unity. Then R is called UFD if

i) Every non-zero, non-unit element a of R can be expressed as Product of finite No. of irreducible elements of R and

ii) if $a = p_1 p_2 \dots p_n$

& $a = q_1 q_2 \dots q_m$

p_i & q_j are irreducible elements in R

then $m = n$ and each p_i associate of some q_j

eg. 1. P.T. $(\mathbb{Z}, +, \cdot)$ is UFD.

Sol $\langle \mathbb{Z}, +, \cdot \rangle$ is I.D with unity

Let $n \in \mathbb{Z}$ non-zero, non-unit Element

i.e. $n \neq 0, \pm 1$

Case I If $n > 1$

then $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, p_i are primes

$n = (p_1 p_1 \dots p_1)(p_2 \dots p_2) \dots (p_r p_r \dots p_r)$

$\Rightarrow n$ is Product of Prime elements of $\mathbb{Z}$.

Every Prime Element is irreducible in $\mathbb{Z}$

$\therefore n$ is Product of irreducible elements of $\mathbb{Z}$.

Case II If $n < -1$ Let $n = -m$, $m > 1$

Let $m = q_1 \, q_2 \cdots q_k$

$n = -m$

$n = (-q_1)(q_2) \cdots q_k$

Hence n is product of Primes and

hence irreducible in Z.

Hence Z is UFD.

example 2. S.T. $Z[\sqrt{-5}]$ is not UFD.

Sol:

$Z(\sqrt{-5})$ is ID with unity.

Now $6 = 2 \cdot 3$

$= (1+\sqrt{-5})(1-\sqrt{-5})$

T.P $2, 3, 1+\sqrt{-5}$ & $1-\sqrt{-5}$ are irreducible elements in $Z\sqrt{-5}$ and 2 is not associate of $1+\sqrt{-5}$ or $1-\sqrt{-5}$.

step I

Let $2 = \alpha\beta$ $\qquad \alpha = a+\sqrt{-5}\,b$

$\qquad \beta = c+\sqrt{-5}\,d$

$2 = (a+\sqrt{-5}\,b)(c+\sqrt{-5}\,d)$

$d(2) = d(a+\sqrt{-5}\,b)(c+\sqrt{-5}\,d)$

$4 = d(a+\sqrt{5}\,i\,b)(c+\sqrt{5}\,i\,d)$

$4 = (a^2+5b^2)(c^2+5d^2)$

Now i) $a^2+5b^2 = 1$ & $c^2+5d^2 = 4$

ii) $a^2+5b^2 = 4$ & $c^2+5d^2 = 1$

iii) $a^2+5b^2 = 2$ & $c^2+5d^2 = 2$.

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i) If $a^2+5b^2 = 1$ & $c^2+5d^2 = 4$

$a = \pm 1,\ b = 0$

$\Rightarrow \alpha = a+\sqrt{5}i\, b$

$\alpha = \pm 1$ a unit

ii) If $a^2+5b^2=4$ & $c^2+5d^2=1$

$c = \pm 1,\ d = 0$

$\beta = c+\sqrt{5}i\, d$

$\beta = \pm 1 \Rightarrow$ a unit.

iii) $a^2+5b^2 = 2$ & $c^2+5d^2 = 2$

have no solution

$\therefore$ 2 is irreducible in $Z[\sqrt{-5}]$.

lly 3 is irreducible in $Z[\sqrt{-5}]$

Step II Let $1+\sqrt{-5} = \alpha\beta$

$= (a+\sqrt{-5}\, b)(c+\sqrt{-5}\, d)$

$d(1+\sqrt{5}i) = d(a+\sqrt{5}i b)\, d(c+\sqrt{5}i d)$

$1^2+5 = (a^2+5b^2)(c^2+5d^2)$

$6 = (a^2+5b^2)(c^2+5d^2)$

i) $a^2+5b^2 = 2$ & $c^2+5d^2 = 3$

ii) $a^2+5b^2 = 3$ & $c^2+5d^2 = 2$

iii) $a^2+5b^2 = 6$ & $c^2+5d^2 = 1$

iv) $a^2+5b^2 = 1$ & $c^2+5d^2 = 6$

i) $a^2+5b^2=2$ & $c^2+5d^2=3$

have no solution in Z.

ii) $a^2+5b^2=3$ & $c^2+5d^2=2$

have no solution in Z.

iii) $a^2+5b^2=1$ & $c^2+5d^2=6$

$a=\pm 1,\ b=0$

$\alpha = a+\sqrt{5}\,i\,d$

$\alpha=\pm 1$ a <u>unit</u>

iv) $a^2+5b^2=6$ & $c^2+5d^2=1$

$c=\pm 1,\ d=0$

$\beta = c+\sqrt{5}\,i\,d$

$\beta=\pm 1$ <u>a unit</u>

$\therefore$ $1+\sqrt{-5}$ is irreducible element in $Z\sqrt{-5}$

||^{ly} $1-\sqrt{-5}$ is irreducible element in $Z\sqrt{-5}$.

Hence Z ~~is~~ not U.F.D